On the zeros of eigenpolynomials of Hermitian Toeplitz matrices

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This talk presents a refinement of a result of Delsarte, Genin, Kamp [2] regarding the number of zeros on the unit circle of eigenpolynomials of complex Hermitian Toeplitz matrices and generalized Caratheodory representations of such matrices. This is achieved by exploring a key observation of Schur [7] stated in his proof of a famous theorem of Carathéodory [1]. In short, Schur observed that companion matrices corresponding to eigenpolynomials of Hermitian Toeplitz matrices H define isometries with respect to (spectrum shifted) submatrices of H. Looking at possible normal forms of these isometries leads directly to the results. This geometric, conceptual approach can be generalized to Hermitian or symmetric Toeplitz matrices over arbitrary fields. Furthermore, as a byproduct, Iohvidov's law in the jumps of the ranks and the connection between the Iohvidov parameter and the Witt index are established for such Toeplitz matrices. In the sequel the topic is explained in more detail.

A famous theorem of Carathéodory [1] states that for arbitrary complex numbers $a_1, ..., a_n$, $n \in \mathbb{N}$ [1], not all zero, there exist uniquely determined data $m \in \mathbb{N}$ with $m \leq n$, pairwise distinct $\varepsilon_1, ..., \varepsilon_m \in \mathbb{C}$ of modulus one and positive real numbers $r_1, ..., r_m$ such that

$$a_i = \sum_{j=1}^{m} r_j \varepsilon_j^i \tag{1}$$

for all $i \in \{1, ..., n\}$. Carathéodory proved his theorem by means of geometric considerations on convex bodies. Soon after Carathéodory, Fischer [3], Schur [7] and also Frobenius [4] gave algebraic proofs of this theorem. From the very beginning the connection between Carathéodory's theorem and Hermitian Toeplitz matrices was clear: Define

$$\mu := \mu(a_1, ..., a_n) := r_1 + ... + r_m > 0,$$

[1] $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$

$R := \mathrm{diag}(r_1, ..., r_m) \in \mathbb{C}^{m,m}$ and the Vandermonde matrix

$$V := \begin{bmatrix} 1 & 1 & \dots & 1 \\ \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_m \\ \varepsilon_1^2 & \varepsilon_2^2 & \dots & \varepsilon_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ \varepsilon_1^n & \varepsilon_2^n & \dots & \varepsilon_m^n \end{bmatrix} \in \mathbb{C}^{n+1,m}. \tag{2}$$

Then, the Hermitian Toeplitz matrix

$$H := H(\mu, a_1, ..., a_n) := \begin{bmatrix} \mu & a_1 & a_2 & \dots & a_n \\ \overline{a_1} & \mu & a_1 & a_2 & \dots \\ \overline{a_2} & \overline{a_1} & \mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \overline{a_n} & \dots & \overline{a_2} & \overline{a_1} & \mu \end{bmatrix} \in \mathbb{C}^{n+1,n+1}.$$

is positive semidefinite and admits the representation $H = \overline{V}RV^T$. Therefore, the vector $p = (p_0, ..., p_m, 0, ..., 0)^T \in \mathbb{C}^{n+1}$ consisting of the coefficients of the polynomial

$$p(x) := \prod_{i=1}^{m}(x - \varepsilon_i) = \sum_{i=0}^{m} p_i x^i \tag{3}$$

is contained in the kernel of H since $V^T p = (p(\varepsilon_1), ..., p(\varepsilon_1)^m)^T = 0$, and $p(x)$ is the uniquely determined nonzero monic (eigen-)polynomial of smallest degree with this property. Conversely, given an arbitrary Hermitian Toeplitz matrix $H = H(a_0, ..., a_n) \in \mathbb{C}^{n+1,n+1}, \quad a_0, ..., a_n \in \mathbb{C}$, which is not a diagonal matrix, i.e., $a_1, ..., a_n$ not all zero, then, necessarily, $\lambda := a_0 - \mu(a_1, ..., a_n)$ is the smallest eigenvalue of H, and, if $m \le n$, ε_i and r_i are chosen according to Carathéodory's theorem for $a_1, ..., a_n$, then H admits the so-called Carathéodory representation

$$H = \overline{V}RV^T + \lambda I_{n+1}\ , \tag{4}$$

where R and V are defined as before. From the previous also follows that $p(x)$ as defined in (3) is the uniquely determined nonzero monic eigenpolynomial of smallest degree corresponding to the smallest eigenvalue λ of H, and is therefore proved to have simple roots on the unit circle. By replacing H by $-H$, the same holds true for the uniquely determined eigenpolynomial of smallest degree corresponding to the largest eigenvalue of H. This special root distribution of eigenpolynomials corresponding to the extremal eigenvalues of Hermitian Toeplitz matrices is widely discussed and repeatedly reproved in the literature, surely also because of its direct applications in the area of signal processing.

A good survey article on this subject was written by Genin [5]. There it is stated that the following result from Delsarte, Genin and Kamp [2] is the most general known one regarding the number of roots on the unit circle of any eigenpolynomial corresponding to any eigenvalue of a given complex Hermitian Toeplitz matrix:

Theorem 1 (Delsarte, Genin, Kamp)**.** *Let $H \in \mathbb{C}^{n+1,n+1}$, $n \in \mathbb{N}_0$, be a Hermitian Toeplitz matrix with eigenvalues $\lambda_0 \leq \lambda_1 \leq ... \leq \lambda_n$, and let λ_s, $s \in \{0, ..., n\}$, be one of them with multiplicity m and Iohvidov parameter k. The index s shall be chosen such that either $s = 0$ or $\lambda_s > \lambda_{s-1}$. Then, any eigenpolynomial $p_s(x)$ corresponding to λ_s has at least $|n - m - 2s + 1|$ and at most $n - 2k$ zeros on the complex unit circle.*

Both bounds stated in Theorem 1 are sharp in the sense that examples exist that attain them. The upper bound $n - 2k$ involves the less commonly known Iohvidov parameter k of an eigenvalue λ of a Hermitian, non-diagonal Toeplitz matrix $H = H(a_0, ..., a_n) \in \mathbb{C}^{n+1,n+1}$. It is defined as follows: Let m be the multiplicity of λ and let $r \in \{0, ..., n\}$ be maximal subject to λ being not an eigenvalue of the principal submatrix $H_r := H(a_0, ..., a_r)$, i.e., $H_r - \lambda I_{r+1}$ is regular. If such an r does not exist, set $r := -1$. It follows from results of Iohvidov [6], that $n - m - r$ is an even, non-negative integer, wherefore the Iohvidov parameter

$$k := \frac{1}{2}(n - m - r) \tag{5}$$

is well-defined. It is known, (see, for example, [2]) that any eigenpolynomial $q(x)$ of H corresponding to λ has the form

$$q(x) = x^k p(x) s(x) \ , \tag{6}$$

where $p(x)$ is the uniquely determined monic eigenpolynomial of degree $r+1$ of H_{r+1} corresponding to the eigenvalue λ and $s(x)$ is an arbitrary polynomial of degree at most $m - 1$. Since $\deg(p(x)s(x)) \leq r + m = n - 2k$, this is clearly an upper bound for the number of zeros on the unit circle of $q(x)$. Thus, actually, only the lower bound $|n - m - 2s + 1|$ given in Theorem 1 is non-trivial.

Delsarte, Genin and Kamp obtain this bound by considering two-variable Levinson polynomials and exploring their properties. The purpose of this talk is to give a different, more conceptual proof, exploiting the observation of Schur [7] that companion matrices of eigenpolynomials of Hermitian Toeplitz matrices define isometries of Hermitian forms defined by certain Toeplitz submatrices.

Looking at the signature of these Hermitian forms and the possible normal forms of those isometries immediately gives the result. Moreover, this approach additionally allows to deduce statements not only on the total number of roots on the unit circle but also on their multiplicities. Recall that also the original result of Carathéodory involves a statement on multiplicities, namely that all roots of the eigenpolynomial of smallest degree corresponding to the smallest/largest eigenvalue of a non-diagonal Hermitian Toeplitz matrix are simple. This fact is by no means trivial compared to just proving that all roots of that eigenpolynomial are located on the unit circle. Furthermore, the perspective of normal forms of Schur's isometries does not rely on the field of complex numbers and draws direction to describing the eigenpolynomial structure of Hermitian or symmetric Toeplitz matrices over arbitrary fields. The main result for the classical complex Hermitian case refining Theorem 1 reads as follows:

Theorem 2 (Main Theorem for complex Hermitian Toeplitz matrices)**.**
Let $H \in \mathbb{C}^{n+1,n+1}$, $n \in \mathbb{N}_0$, be a Hermitian Toeplitz matrix with eigenvalues $\lambda_0 \le \lambda_1 \le ... \le \lambda_n$, and let $\lambda = \lambda_s$, $s \in \{0, ..., n\}$, be one of them with multiplicity m and Iohvidov parameter k. The index s shall be chosen such that either $s = 0$ or $\lambda_s > \lambda_{s-1}$. Furthermore, let $p(x)$ be the monic eigenpolynomial of smallest degree corresponding to λ. The distinct roots of $p(x)$ on the unit circle are denoted by $\alpha_1, ..., \alpha_a$, $a \in \mathbb{N}_0$, and their multiplicities by $m_1, ..., m_a$. The remaining non-zero roots of $p(x)$ occur in conjugate pairs $\{\beta_1, \overline{\beta_1}^{-1}\}, ..., \{\beta_b, \overline{\beta_b}^{-1}\}$, $b \in \mathbb{N}_0$, where β_i and $\overline{\beta_i}^{-1}$ have the same multiplicity n_i, $i = 1, ..., b$. Then,

$$a \ge |\{i \in \{1, ..., a\} \mid m_i \textit{ odd }\}| \ge |n - m - 2s + 1| \ , \tag{7}$$

$$\sum_{i=1}^{a} \left\lfloor \frac{m_i}{2} \right\rfloor + \sum_{i=1}^{b} n_i \le \frac{n - m + 1 - |n - m - 2s + 1|}{2} - k \ . \tag{8}$$

In the extremal cases $s = 0$ (smallest eigenvalue) and $s = n - m + 1$ (largest eigenvalue) holds $\frac{n-m+1-|n-m-2s+1|}{2} = 0$ wherefore (8) implies $k = 0 = b$, and $m_i = 1$ for all $i = 1, ..., a$, i.e., all roots of $p(x)$ lie on the unit circle and are simple. This is the classical result following from Carathéodory's theorem. Note also, that by Equation (6) Theorem 2 gives (sharp) lower bounds for the multiplicities of the roots on the unit circle of any eigenpolynomial.

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