Spherical Designs and Polynomial Approximation on the Sphere

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This talk presents some joint work with An, Frommer, Lang, Sloan and Womersley on spherical designs and polynomial approximation on the sphere [1],[2],[4],[5].

Finding "good" finite sets of points on the unit sphere S^d in the Euclidean space R^{d+1} has been a hot research topic in mathematics, physics, and engineering for more than hundred years. There are several concepts of "good" finite sets of points on S^d for different purposes. A *spherical t-design* is considered a "good" finite set of points on S^d for global polynomial approximations on the sphere.

A finite set $Y = \{y_1, \ldots, y_N\} \subset S^d$ is called a spherical t-design on S^d if for any polynomial $p : R^{d+1} \to R$ of degree at most t, the average value of p on the set equals the average value of p on the whole sphere, that is,

$$\frac{1}{N}\sum_{i=1}^{N} p(y_i) = \frac{1}{|S^d|}\int_{S^d} p(y)dw(y), \tag{1}$$

where $|S^d|$ is the surface of the whole unit sphere S^d and $dw(y)$ denotes the surface measure on S^d.

The concept of a spherical t-design was introduced by Delsarte, Goethals and Seidel [6] in 1977. The existence of a spherical t-design for any $t \geq 1$ and $d \geq 1$ was proved by Seymour and Zaslavsky in 1984 [8]. However, there is no known answer to the question of the number of points N needed to construct a spherical t-design for any $t \geq 1$ and $d \geq 1$.

For the case $d = 2$, finding spherical t-designs has many applications. The earth's surface is an approximate sphere S^2, and spherical t-designs are relevant to many problems of geophysics, including climate modeling and global navigation. Moreover, polynomial approximation on S^2 has wide applications in coding communications, viruses analysis and molecular chemistry. In this talk, we focus our study on spherical t-designs for S^2.

Let P_t be the linear space of restrictions of polynomials of degree $\leq t$ in 3 variables to S^2. Since the surface of the whole unit sphere S^2 is 4π, the equality (1) for S^2 can be written as

$$\int_{S^2} p(y)dw(y) = \frac{4\pi}{N}\sum_{i=1}^{N} p(y_i), \quad \text{for all } p \in P_t. \tag{2}$$

A lower bound on the number of points N_t needed to construct a spherical t-design for any $t \geq 1$ and $d = 2$ was given in [6]:

$$N_t \geq N_t^* = \begin{cases} \frac{1}{4}(t+1)(t+3) & \text{if } t \text{ is odd} \\ \\ \frac{1}{4}(t+2)^2 & \text{if } t \text{ is even.} \end{cases}$$

However, it is shown in [6] that the lower bound can not be achieved, that is, there is no spherical t-design with N_t^* points for any $t \geq 2$.

Hardin and Sloane [7] proposed a sequence of putative spherical t-designs with $\frac{1}{2}t^2 + o(t^2)$ points and presented numerical spherical t-designs for $t \leq 21$. Sloan and Womersley [9] established a new variational characterization of spherical designs: it is shown that a set $Y = \{y_1, \dots, y_N\} \subset S^2$ is a spherical t-design if and only if a certain non-negative quantity vanishes. Using their characterization, Sloan and Womersley numerically obtained spherical t-designs for $t \leq 19$.

The dimension of the space P_t is $d_t := (t+1)^2$. A set of points $Y = \{y_1, \dots, y_{d_t}\}$ is called a *fundamental system* if the zero polynomial is the only member of P_t that vanishes at each point $y_j, j = 1, \dots, d_t$, which is equivalent to the $(t+1)^2 \times (t+1)^2$ Gram matrix at these points being nonsingular. Let $\mathcal{Y}$ denote the set of all fundamental systems. Chen and Womersley [5] presented a characterization of fundamental spherical t-designs: it is shown that a system $Y = \{y_1, \dots, y_N\} \in \mathcal{Y}$ is a spherical t-design if and only if $Y \in \mathcal{Y}$ is a solution of a certain system of nonlinear equations. Chen and Womersley numerically computed approximate solutions for this system of nonlinear equations. They then numerically checked that the hypothesis of an appropriate generalization of the Newton–Kantorovich theorem holds. In this manner, they proved the existence of an exact solution of the nonlinear system close to the numerically computed approximation—and thus the existence of a t-design with d_t points. This approach is quite expensive computationally and was thus carried out for $t \leq 20$, only. Also, the hypothesis check for the Newton–Kantorovich type theorem was done using floating point arithmetic, so that numerical rounding prevents this approach from providing a proof in a strict mathematical sense.

It is known that finding spherical t-designs for large t is very difficult [3]. Up to now, spherical t-designs for $t \geq 21$ have not been verified. Evaluating large scale polynomials of high degree exactly or with known tight error bounds is a challenging problem in practical computation. In [4], Chen, Frommer and Lang use the characterization of fundamental spherical t-designs in [5] and interval arithmetic to find exact spherical t-designs with d_t points for t up to 100. Since also the effects of floating point rounding are completely accounted for, this computational approach has the quality of a "true" mathematical proof. In particular, Chen, Frommer and Lang compute tight bounds for a set of points $Y = \{y_1, \dots, y_{(t+1)^2}\} \subset S^2$ which is proved to be a fundamental system as well as a solution of the system of nonlinear equations from [5] which makes it a spherical t-design.

Examples show that just being a solution of the system of nonlinear equations is not sufficient to be a spherical t-design. So we really have to verify that the set of points is not only a solution of the system of nonlinear equations, but also a fundamental system.

In this talk, first we describe the characterization of fundamental spherical t-designs from [5] and our approach to find a solution of the system of nonlinear equations. Next we recall Krawczyk's method which uses interval arithmetic to provide narrow intervals for each component of a vector which provably contains a solution of the system of nonlinear equations. We also show how one can use interval arithmetic to prove the nonsingularity of the Gram matrix over this interval vector. This then shows that the solution contained in the interval vector is also a fundamental system and thus a t-design. Moreover, the radii of the intervals enclosing the components are at most $\sim 10^{-9}$ for t up to 100, so the coordinates of the points of the t-design are known to high accuracy. We present our numerical results which prove the existence of t-designs with d_t points for t up to 100. So our work provides strong evidence that spherical t-designs with $(t+1)^2$ points will probably exist for any t. In this sense, this paper further contributes to the solution of the open problem on the minimal number of points needed to construct a spherical t-design. Moreover, our results suggest that for any $t \geq 1$, there is a *fundamental* spherical t-design.

Finally, we report some results on polynomial approximation on the unit sphere S^2 by a class of regularized discrete least squares methods with novel choices for the regularization operator and spherical t-design. We allow different kinds of rotationally invariant regularization operators, including the zero operator (in which case the approximation includes interpolation, quasi-interpolation, and hyperinterpolation); powers of the negative Laplace. Beltrami operator (which can be suitable when there are data errors); and regularization operators that yield filtered polynomial approximations. For $t \geq 2L$ and an approximating polynomial of degree L it turns out that there is no linear algebra problem to be solved and the approximation in some cases recovers known polynomial approximation schemes, including interpolation, hyperinterpolation, and filtered hyperinterpolation. For $t \in [L, 2L)$ the linear system needs to be solved numerically. Finally, we give numerical examples to illustrate the theoretical results and show that well chosen regularization operator and well conditioned spherical t-designs can provide good polynomial approximation on the sphere, with or without the presence of data errors.

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