# Exceptional surgeries on alternating knots


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A *3-dimensional manifold*, often abbreviated by *3-manifolds*, is defined as a topological space locally modeled on the 3-dimensional Euclidean space $\mathbb{R}^3$. For example, our universe seems to be a 3-dimensional space locally near our earth, and so, if we believe the cosmological principle, i.e., viewed on a sufficiently large scale, the properties of the universe are the same for all observers, then our universe is an example of a 3-manifold.

The study of 3-dimensional manifolds starts with the seminal papers by H. Poincaré in 1894–1904. In the last of these six papers, he raised a question, which is now called the *Poincaré Conjecture*; *Is every simply connected, closed 3-manifold homeomorphic to the 3-sphere?* (The 3-sphere $S^3$ is the most simple closed 3-manifold, which can be regarded as $\mathbb{R} \cup \{\infty\}$, i.e., the one-point compactification of $\mathbb{R}^3$.) This has been one of the driving forces for (low-dimensional) topology, and a great amount of effort was spent in pursuit of a solution. In fact, it was selected as one of the Millennium Prize Problems stated by the Clay Mathematics Institute in 2000.

After nearly a century later, in 2002–2003, G. Perelman finally reached to the end of the struggles by providing an affirmative answer to the Geometrization Conjecture, an extended version to the Poincaré Conjecture. See [7] as detailed references for example.

By the celebrated Perelman's works, we now have a classification theorem for 3-manifolds. Beyond the classification theorem, one of the next directions in the study of 3-manifolds is to consider the relationships between 3-manifolds. One of the important operations describing such a relationship would be *Dehn surgery*; an operation to create a new 3-manifold from a given one and a given knot by removing an open tubular neighborhood of the knot, and gluing a solid torus back. This gives an interesting subject to study; because, for instance, it is known that any pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries on knots.

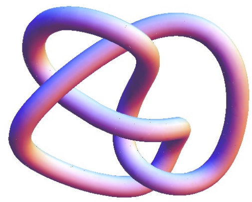
On the other hand, by the classification theorem for 3-manifolds, now we know that all closed orientable 3-manifolds are classified into; *reducible* (i.e., containing essential 2-spheres), *toroidal* (i.e., containing essential tori), *Seifert fibered* (i.e., foliated by circles), or *hyperbolic* manifolds (i.e., admitting a complete Riemannian metric with constant sectional curvature $-1$).

Among the above four classes of 3-manifolds, many researchers would believe that the hyperbolic 3-manifolds are "ubiquitous" in a sense. This intuition can be justified in terms of Dehn surgery as follows.

The well-known Hyperbolic Dehn Surgery Theorem says that, all but only finitely many Dehn surgeries on a hyperbolic knot (i.e., a knot with the complement admitting a hyperbolic structure) yield hyperbolic manifolds. In view of this, such finitely many exceptions are called *exceptional surgeries*.

To establish a complete classification of exceptional surgeries on hyperbolic knots in the 3-sphere $S^3$ would be one of the most important but challenging problems in the study of 3-manifolds, and also in Knot Theory. Toward the ultimate goal to this problem, some of the partial solutions have been obtained.

Here we consider exceptional surgery on hyperbolic alternating knots in $S^3$, one of the most well-known classes of knots. A knot in $S^3$ is called *alternating* if it admits a diagram with alternatively arranged over-crossings and under-crossings running along it. See the right figure for example. (The knot is so-called the *figure-eight knot*, that is the simplest hyperbolic knot, i.e., the complement has the minimal volume among hyperbolic knots.)



Our main result is to provide a complete classification of exceptional surgeries on alternating knots. We here omit the details. Please see our paper [3].

We insist that our result is purely mathematical, but our proof is computer-aided via verified numerical computation. Actually, due to the result of Lackenby [5], we have only finitely many (but a huge number of) links to be checked. Thus our task is to investigate the surgeries on these finite number of links.

In [6], Martelli-Petronio-Roukema gave a complete classification of exceptional surgeries on the minimally twisted five-chain link, a well-known hyperbolic link of 5 components in $S^3$, via a computer-aided method. Our program is essentially due to their technique, but, to achieve mathematically rigorous computations, we improved their codes using verified numerical analysis based on interval arithmetics. However applying it for all the links obtained by [5] is computationally expensive, for the number of the links are roughly estimated to be in the millions. Therefore we give a number of (mathematical) observations to reduce the number of links we need to check. Even with this reductions, as the verification needed for each link is an involved process, the size of the computation is outside the scope of a personal computer. To be more specific, we have 30,404 links to investigate, and for each single link, we have to apply recursively the program "hikmot" [2], developed in [1] via verified numerical analysis. In fact, in the worst case, we have to apply the procedure more than 18,000 times for a link. Therefore, we ran our computations on the super-computer, "TSUB-

AME" (Tokyo-tech Supercomputer and UBiquitously Accessible Mass-storage Environment), housed at Tokyo Institute of Technology. See its website [8] for a basic information. Roughly speaking, on TSUBAME, one can use many machines at the same time. Although generally, to use parallel computation effectively we need some work, in our case, the situation itself is totally parallel, that is, we need to investigate each link independently. Thus we can use TSUBAME effectively. In practice, we "rent" 320 machines from TSUBAME, and then it took a day to achieve our result. By running our main program `fef.py` (short for find exceptional fillings) on TSUBAME, we have 30404 output files and error files. A pseudo-code for `fef.py` is provided as Algorithm 1, and all codes and the result data are available at [4].

In the worst case, it takes about 51 hours on a single CPU of TSUBAME (The computational ability of a single CPU of TSUBAME is comparable to that of a standard personal computer). In total, i.e. the sum of the computation time of all nodes, computation time was approximately 512 days, and the number of manifolds we applied `hikmot` is 5,646,646. Fortunately, for all 30404 links, the outputs show that they have no unexpected exceptional surgery. This verifies our proof of the theorem.

**References:**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[3] K. Ichihara, H. Masai, Exceptional surgeries on alternating knots, preprint, arXiv:1310.3472

[4] K. Ichihara, H. Masai, `http://www.math.chs.nihon-u.ac.jp/~ichihara/ExcAlt/index.html` where *fef.py* and *twistlink.py's* are available.

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**Algorithm 1** The algorithm for fef.py

**Require:** A triangulation $T$ of a manifold $N$.

**Ensure:** A verification that all non-trivial Dehn surgeries of a manifold fitting certain conditions are hyperbolic.

Try to canonize $T$.

**if** T can be canonized and hikmot verified the hyperbolicity of canonized triangulation. **then**

Use the canonized triangulation.

**else if** Find a triangulation whose hyperbolicity is checked by hikmot. **then**

Use the found triangulation.

**else**

If we cannot find any triangulation that hikmot verifies hyperbolicity, we give up. (This didn't happen in our computation for alternating knots)

**end if**

Compute lower bounds for the cusped areas of $N$ using the (already) verified tetrahedral shapes for $T$. For each cusp, also compute the cusp shape as an parallelogram determined by a quotient of the complex plane by 1 and $x + yi$. Finally, compute a lower bound for the diameter of the horoball for that cusp and enforce with this bound that the intersection of the boundary of a horoball (not centered at $\infty$) and a ideal tetrahedron having a vertex at $\infty$ intersect in a triangle.

**if** Failed on some procedure above. **then**

Use $\frac{3\sqrt{3}}{8}$ as a lower bound for cusp area. For these cusps, the cusp shape is determined by 1 and $x + yi$ with $x = 0$ and $y = \frac{3\sqrt{3}}{8}$.

**end if**

The length of a slope $\frac{p}{q}$ is $\sqrt{\frac{A}{y}((p + xq)^2 + (yq)^2}$, where $A$ is the area of corresponding horosphere. List all slopes of length less than 6.0001 in these cusps. For slopes on each cusp less than length 6.0001, perform surgery with that slope if it meets certain conditions.

**if** All cusps have been surgered along. **then**

Verify that the surgered manifold is hyperbolic.

**else**

Verify this intermediately surgered manifold is hyperbolic and repeat the procedure above to find all slopes of length less than 6.0001 in the cusps of this partially surgered manifold and (recursively) verify the hyperbolicity of these surgeries.

**end if**