Validated solutions for P-matrix linear complementarity problems

Takuma Kimura

Faculty of Science and Engineering, Waseda University / JST CREST,
Building 63, Room 419, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan.

tkimura@aoni.waseda.jp

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The linear complementarity problem (LCP),

$$Mx + q \geq 0, \quad x \geq 0, \quad x^T(Mx + q) = 0,$$

can be reformulated as the following system of piecewise linear equations,

$$\min(x,\, Mx + q) = 0, \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$, $q \in \mathbb{R}^n$.

It is known that (1) has a unique solution for any vector q if and only if M is a P-matrix, i.e. M satisfies the condition, $\max\limits_{1\leq i\leq n} v_i(Mv)_i > 0$, for all $v \neq 0$ (Theorem 3.3.7 in [2]). We assume that M is a P-matrix.

LCP has many applications in engineering and economics. Then, many numerical methods for solving the LCP have been proposed. Therefore, it is important to verify the accuracy of the solutions obtained by numerical methods.

Let x^* be an exact solution and x be an approximate solution of (1). We define the natural residual function,

$$r(x) = \min(x, Mx + q).$$

In [3], Chen and Xiang proposed an error bound in $\|\cdot\|_p$ ($p \geq 1$ or $p = \infty$),

$$\|x - x^*\|_p \leq \beta_p(M)\|r(x)\|_p, \tag{2}$$

where

$$\beta_p(M) = \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_p.$$

For some special matrices, we can compute $\beta_p(M)$ easily, for example, if M is an M-matrix then $\beta_p(M) = \|M^{-1}\|_p$. Moreover, Chen and Xiang proved that (2) is sharper than the Mathias-Pang error bound [4] when $p = \infty$. However, for general P-matrix, we do not have simple upper bound for $\beta_p(M)$.

Here, we consider an equation in the proof of (2). Let $y = Mx + q$ and $y^* = Mx^* + q$. For any $x, x^* \in \mathbb{R}^n$, we have

$$r(x) = (I - D + DM)(x - x^*), \tag{3}$$

where D is a diagonal matrix whose diagonal elements are d_i,

$$d_i = \begin{cases} 0, & y_i \geq x_i,\, y_i^* \geq x_i^*, \\ 1, & y_i \leq x_i,\, y_i^* \leq x_i^*, \\ \dfrac{\min(x_i, y_i) - \min(x_i^*, y_i^*) - x_i + x_i^*}{y_i - y_i^* - x_i + x_i^*}, & \text{otherwise.} \end{cases} \tag{4}$$

Moreover, we have $d_i \in [0, 1]$. It is known that M is a P-matrix if and only if $I - D + DM$ is nonsingular for any diagonal matrix D =diag(d) with $d \in [0, 1]^n$ [2].

In this talk, we consider a componentwise numerical verification algorithm based on (3). Applying the computable rough error bounds (e.g., Ex. 5.11.20 in [2]), we can check the magnitude relations between x, y, x^*, y^*. Then, from (4), we can compute an interval vector contains d. If we have sufficiently tight interval vector contains d then we can compute an nonsingular interval matrix contains $I - D + DM$ and the componentwise error bound by solving systems of interval linear equations derived from (3).

More details, examples and comparison with existing methods [3, 6, 7, 8] will be shown in the talk.

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