Verified lower eigenvalue bounds for self-adjoint differential operators

[1]Xuefeng Liu and [2]Shin'ichi Oishi

[1]Research Institue of Science and Engineering, Waseda University;
[2]Faculty of Science and Engineering, Waseda University; CREST/JSP

xfliu@aoni.waseda.ac.jp

Keywords: Verified eigenvalue bounds, finite element method, self-adjoint differential operators

By using finite element methods(FEM), we develop a theorem to give verified eigenvalue bounds for generally defined self-adjoint differential operators, which includes the Laplace operator, the Biharmonic operators and so on. The explicit a priori error estimations for conforming and non-conforming FEMs play import role in constructing explicit lower eigenvalue bounds. As a feature of proposed theorem, it can even give bounds for eigenvalues that the corresponding eigenfunctions may have a singularity. The verified eigenvalue bounds can be applied to give explicit bounds for various interpolation error constants and help to verify the solution existence for non-linear partial differential equations.

We consider the eigenvalue problem in an abstract form. Let V be a Hilbert function space and V^h be a finite dimensional space, $\mathrm{Dim}(V^h)=n$. Here, V^h may not be a subspace of V. Suppose $M(\cdot,\cdot)$, $N(\cdot,\cdot)$ are semi-positive symmetric bilinear forms on both V and V^h. Moreover, for any $u\in V$ or V^h, $N(u,u)\geq 0$ implies $u=0$. Define norm $|\cdot|_N$ and semi-norm $|\cdot|_M$ by, $|\cdot|_M := \sqrt{M(\cdot,\cdot)}$, $|\cdot|_N := \sqrt{N(\cdot,\cdot)}$. We consider an eigenvalue problem defined by the bilinear forms $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$: Find $u\in V$ and $\lambda \in R$ such that,

$$M(u,v)=\lambda N(u,v) \quad \forall v\in V. \tag{1}$$

Further we take the following two basic assumptions for the eigen-pair of (1):

1. The eigenvalues can be ordered as an ascending sequence: $0\leq \lambda_1 \leq \lambda_2 \cdots$.
2. The eigenfunctions $\{u_i\}_{i=1}^{\infty}$ form a complete orthonormal basis of V, i.e.,

$$u=\sum_{i=1}^{\infty}(u,u_i)u_i, \text{ for any } u\in V; \quad N(u_i,u_j)=\delta_{ij} \quad (\delta_{ij}: \text{ Kronecker's delta }).$$

Let $(\lambda_{h,k}, u_{h,k})$ $(1\leq k\leq n)$ be the eigen-pair of the eigenvalue problem over finite dimensional space V^h: Find $u_h\in V^h$ and $\lambda_h\in R$ such that,

$$M(u_h,v_h)=\lambda_h N(u_h,v_h) \quad \forall v_h\in V^h. \tag{2}$$

The main theorem to provide lower eigenvalue bounds is given as below.

Theorem 1. *Let $P_h : V \to V^h$ be a projection such that,*

$$M(u - P_h u, v_h) = 0 \quad \forall v_h \in V^h \ , \tag{3}$$

along with an error estimation as

$$|u - P_h u|_N \le C_h |u - P_h u|_M \ . \tag{4}$$

Then we have lower bounds for λ_k's,

$$\frac{\lambda_{h,k}}{1 + \lambda_{h,k} C_h^2} \le \lambda_k \quad (k = 1, 2, \cdots, n) \ . \tag{5}$$

The concrete form of projection P_h and the value of constant C_h, which tends to 0 as mesh size $h \to 0$, are depending on the FEM spaces in use. In case of the eigenvalue problems of Laplacian, a new method based on hyper-circle equation for conforming FEM spaces is developed to give an explicit bound for the constant C_h ; see [1]. Generally, by using proper non-conforming FEMs, the projection P_h is just an interpolation operator and the constant C_h can be easily obtained by considering the interpolation error estimation on local elements. Such a new result will be reported in the workshop.

Usually, Theorem 1 along with finite element methods can only provide rough eigenvalue bounds for objective differential operators; see discussion in [4]. For purpose of high-precision eigenvalue evaluation, we combine rough lower eigenvalue bounds and Lehmann-Goerisch's theorem to give sharp bounds. Such method has been used to give high-precision bounds for several interpolation error constants; see [2]. Also, the explicit a priori error estimation for FEMs has been successfully applied to verify the solution existence for semi-linear elliptic partial differential equations [3].

References:

[1] Xuefeng Liu and Shin'ichi Oishi, Verified eigenvalue evaluation for Laplacian over polygonal domain of arbitrary shape, *SIAM J. Numer. Anal.*, 51(3), 1634-1654, 2013.

[2] Xuefeng Liu and Shin'ichi Oishi, Guaranteed high-precision estimation for P_0 interpolation constants on triangular finite elements, *Japan Journal of Industrial and Applied Mathematics*, 30(3), 635-652, 2013.

[3] Akitoshi Takayasu, Xuefeng Liu and Shin'ichi Oishi, Verified computations to semilinear elliptic boundary value problems on arbitrary polygonal domains, *NOLTA, IEICE*, Vol.E96-N, No.1, pp.34-61, Jan. 2013.

[4] A Takayasu, X Liu and S Oishi, Remark on computable a priori error estimation for higher degree finite element solution of elliptic problem, *NOLTA, IEICE*, Vol.5, No.1, pp.53-63, Jan. 2014.