# A method of verified computations for nonlinear homogeneous heat equations,

## Part II: Semigroup approach to construct an exact solution for time variable

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Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. In this talk, we consider a method of verified computations for nonlinear heat equations of the form:

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } (0,\infty) \times \Omega, \\ u|_{\partial\Omega} = 0 & \text{in } (0,\infty), \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where $f : L^\infty((0,\infty); H_0^1(\Omega)) \to L^\infty((0,\infty); L^2(\Omega))$ be a nonlinear mapping. We assume the Fréchet differentiable of $f$ with respect to $u$ on the spatial direction. Furthermore, let $u_0 \in D(\Delta)$ be a given initial function. Let $n \in \mathbb{N}$ be a fixed natural number. We divide the time: $0 = t_0 < t_1 < \cdots < t_n < \infty$. For $k = 1, 2, ..., n$, we define $T_k := (t_{k-1}, t_k]$, $\tau_k := t_k - t_{k-1}$, and $T := \bigcup T_k$. We can compute $\hat{u}_k \approx u(t_k)$ using the scheme established in [1]. Then we construct an approximate solution of (1), which is denoted by $\omega(t) \in L^\infty(T; H_0^1(\Omega))$:

$$\omega(t) := \sum_{k=1}^{n} \hat{u}_k \phi_k(t), \ t \in T,$$

where $\phi_k(t)$ is piecewise linear Lagrange basis on each $T_k$ defined by

$$\phi_k(t) := \begin{cases} \dfrac{t - t_{k-1}}{t_k - t_{k-1}}, & t \in T_k, \\ \dfrac{t_{k+1} - t}{t_{k+1} - t_k}, & t \in T_{k+1}, \\ 0, & \text{otherwise}. \end{cases}$$

In this part, we introduce how to construct explicitly $u(t)$ of (1) on the basis of the ideal approximate solution $\bar{u}(t) \in W^{1,1}(T; H_0^1(\Omega))$ defined by

$$\bar{u}(t) := \sum_{k=1}^{n} u_k \phi_k(t), \ t \in T.$$

By using several techniques [2,3,4,5] of verified computations for elliptic problems, we can enclose the ideal approximate solution based on $\omega(t)$. For a given $v \in L^\infty(T_k; H_0^1(\Omega))$ and $\rho > 0$, we define a closed ball $B_k(v, \rho)$ centered at $v$ with radius $\rho$. Let us assume that it satisfies

$$L_k(v, \rho) := \sup_{y \in B_k(v,\rho), t \in T_k} \|f'[y(t)]\|_{V,X} < \infty$$

Then we have the following theorem on each time interval $T_k$.

**Theorem 1.** *Let $\lambda_{\min}$ be the minimal eigenvalue of the Laplacian with homogeneous boundary condition. We set*

$$\alpha := \left( 2\sqrt{\frac{\tau_k}{e}} L_k(\bar{u}, \|u_k - u_{k-1}\|_V) + 1 + \frac{1 - e^{-\tau_k \lambda_{\min}}}{\tau_k \lambda_{\min}} \right) \|u_k - u_{k-1}\|_V .$$

*If $\beta_k > 0$ satisfies*

$$\left( 1 - 2\sqrt{\frac{\tau_k}{e}} L_k(\bar{u}, \beta_k) \right) \beta_k - \alpha \geq 0 \ \textit{and} \ \sqrt{\frac{\tau_k}{e}} L_k(2\bar{u}, \beta_k) < \frac{1}{2},$$

*then the weak solution $u(t)$ of (1) uniquely exits in $B_k(\bar{u}, \beta_k)$ for $t \in T_k$.*

The proof of theorem 1 will be introduced in the talk. The semigroup theory becomes a key part of the proof based on Banach's fixed point theorem. Furthermore, the convergence ratio of our estimates will be shown using some numerical experiments.

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