A Modified Verification Method for Linear Systems

Atsushi Minamihata[1], Kouta Sekine[1], Takeshi Ogita[3,5], Siegfried M. Rump[4], Shin'ichi Oishi[2,5]

[1] Graduate School of Fundamental Science and Engineering, Waseda University

[2] Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University
3–4–1 Okubo, Shinjuku-ku, Tokyo 169–8555, Japan

[3] School of Arts and Sciences, Tokyo Woman's Christian University
2–6–1 Zempukuji, Suginami-ku, Tokyo 167–8585, Japan

[4] Institute for Reliable Computing, Hamburg University of Technology
Schwarzenbergstraße 95, Hamburg 21071, Germany

[5] JST, CREST

aminamihata@moegi.waseda.jp

Keywords: Verified Numerical Computations, Linear Systems

This talk is concerned with the problem of verifying the accuracy of approximate solutions of linear systems. Let $\mathbb{R}$ be the set of real numbers. For $A \in \mathbb{R}^{n\times n}$, the comparison matrix of A is denoted by $\langle A \rangle$.

For a linear system

$$Ax = b, \quad A \in \mathbb{R}^{n\times n},\ b \in \mathbb{R}^n, \tag{1}$$

we can efficiently obtain a computed solution by some numerical algorithm. In general, however, we do not know how accurate the computed solution is. For this purpose, several verification methods have been proposed (cf. e.g. [1,2]).

In this talk we propose an efficient method of calculating a componentwise error bound of the computed solution of (1), which is based on the following Rump's theorem:

Theorem 1 (Rump [1, Theorem 2.1])**.** *Let $A \in \mathbb{R}^{n\times n}$ and $b, \tilde{x} \in \mathbb{R}^n$ be given. Assume $v \in \mathbb{R}^n$ with $v > \mathbf{0}$ satisfies $u := \langle A \rangle v > \mathbf{0}$. Let $\langle A \rangle = D - E$ denote the splitting of $\langle A \rangle$ into the diagonal part D and the off-diagonal part $-E$, and define $w \in \mathbb{R}^n$ by*

$$w_k := \max_{1\le i\le n} \frac{G_{ik}}{u_i} \quad \text{for } 1 \le k \le n,$$

where $G := I - \langle A \rangle D^{-1} = ED^{-1} \ge O$. Then A is nonsingular, and

$$|A^{-1}b - \tilde{x}| \le (D^{-1} + vw^T)|b - A\tilde{x}|. \tag{2}$$

In particular, the method based on Theorem 1 is implemented in the routine `verifylss` in INTLAB Version 7 [3].

We modify Theorem 1 as follows:

Theorem 2. *Let $A, b, \tilde{x}, u, v, w$ be defined as in Theorem 1. Define $D_s :=$ $\mathrm{diag}(s)$ where $s \in \mathbb{R}^n$ with*

$$s_k := u_k w_k \quad \text{for } 1 \le k \le n.$$

Then

$$|A^{-1}b - \tilde{x}| \le (D^{-1} + vw^T)(I + D_s)^{-1}|b - A\tilde{x}|. \tag{3}$$

Proof. From the definition of u and w, it holds

$$I - \langle A \rangle D^{-1} \le uw^T.$$

Since $\mathrm{diag}(I - \langle A \rangle D^{-1}) = \mathbf{0}$, we have

$$I - \langle A \rangle D^{-1} + D_s \le uw^T$$

and

$$I + D_s \le \langle A \rangle D^{-1} + uw^T. \tag{4}$$

From the assumption, A is an H-matrix, so that $\langle A \rangle^{-1} \ge O$. Multiplying (4) from the left by $\langle A \rangle^{-1}$ yields

$$\langle A \rangle^{-1}(I + D_s) \le D^{-1} + \langle A \rangle^{-1} uw^T = D^{-1} + vw^T.$$

Since $(I + D_s)^{-1} \ge O$, we have

$$\langle A \rangle^{-1} \le (D^{-1} + vw^T)(I + D_s)^{-1}. \tag{5}$$

Using $|(A)^{-1}| \le \langle A \rangle^{-1}$ and (5),

$$\begin{aligned} |A^{-1}b - \tilde{x}| &\le |A^{-1}||b - A\tilde{x}| \le \langle A \rangle^{-1}|b - A\tilde{x}| \\ &\le (D^{-1} + vw^T)(I + D_s)^{-1}|b - A\tilde{x}|, \end{aligned}$$

which proves the theorem. □

Theorem 2 always gives better bounds than Theorem 1 since $(I + D_s)^{-1} \le I$. Numerical results will be shown to illustrate the efficiency of the proposed theorem. A generalization of Theorem 2 will be presented.

References:

[1] S. M. Rump, Accurate solution of dense linear systems, Part II: Algorithms using directed rounding, *J. Comp. Appl. Math.*, 242 (2013), 185–212.

[2] S. Oishi, S. M. Rump, Fast verification of solutions of matrix equations, *Numer. Math.*, 90:4 (2002), 755–773.

[3] S. M. Rump, INTLAB - INTerval LABoratory, *Developments in Reliable Computing*, T. Csendes, ed., 77–104, Kluwer, Dordrecht, 1999. `http://www.ti3.tuhh.de/rump/`