Numerical verification method of solutions for nonlinear elliptic and evolutional problems

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1 Elliptic problems

We briefly describe the basic principles for the numerical verification of solutions to the following elliptic problems, see [2],[6] for details,

$$\begin{cases} -\Delta u &= f(x,u,\nabla u), \quad x \in \Omega, \\ u &= 0 \qquad x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n $(1 \leq n \leq 3)$, f is a nonlinear map. We use the homogeneous Sobolev space $H_0^1(\Omega)(\equiv H_0^1)$ for the solution of (1). Also some appropriate assumptions are imposed on the map f. In order to treat the problem as the finite procedure, we use a finite element subspace S_h of H_0^1 with mesh size h.
Denoting the inner product on $L^2(\Omega)$ by $(\cdot,\cdot)$, we define the H_0^1-projection: $P_h\phi \in S_h$ for $\phi \in H_0^1$, by

$$(\nabla\phi - \nabla(P_h\phi), \nabla v_h) = 0, \quad \forall v_h \in S_h. \tag{2}$$

If $\Delta\phi \in L^2(\Omega)$, then the following error estimates plays an essential role to bridge between the infinite and finite dimensional, i.e., continuous and discrete, problems.

$$||(I - P_h)\phi||_{H_0^1} \leq C(h)||\Delta\phi||_{L^2}. \tag{3}$$

Here, I stands for the identity on H_0^1 and $C(h)$ means a positive constant which can be numerically determined such that $C(h) \to 0$ if $h \to 0$. For each $\psi \in L^2(\Omega)$, we denote a solution $\phi \in H_0^1$ of the Poisson equation : $-\Delta\phi = \psi$ with homogeneous boundary condition by $\phi \equiv (-\Delta)^{-1}\psi$. Then, under some appropriate conditions on f, (1) is rewritten as the fixed point equation of the form $u = F(u)$ with a compact map $F \equiv (-\Delta)^{-1}f$ on H_0^1.
The following decomposion of the fixed point equation gives an essential principle which enables us to treat the problem by finite procedure on computer.

$$\begin{cases} P_h u &= P_h F(u) \\ (I - P_h)u &= (I - P_h)F(u) \end{cases} \tag{4}$$

Here, the first and second parts can be considered as equations in S_h and in the orthogonal complement $S_h^\perp$ of H_0^1, respectively.

Sequential iterative method A set $U \subset H_0^1$ which possibly includes a solution of (4) is called *a candidate set* of solutions. Usually, for the sets $U_h \subset S_h$ and $U_\perp \subset S_h^\perp$, the candidate set U is taken as $U = U_h \oplus U_\perp$. Then, a verification condition based on Schauder's fixed point theorem is given by

$$\begin{cases} P_h F(U) & \subset \quad U_h \\ (I - P_h)F(U) & \subset \quad U_\perp \end{cases} \tag{5}$$

The set U_h is taken to be a set of linear combinations of basis functions in S_h with interval coefficients, while $U_\perp$ a ball in $S_h^\perp$ with radius $\alpha \geq 0$.
Note that, it can be easily seen that $P_h F(U)$ is directly computed or enclosed for given U_h and $U_\perp$ by solving a linear system of equations with interval right-hand side using some interval arithmetic approaches. On the other hand, $(I - P_h)F(U)$ can be evaluated as a positive real number by the use of constructive a priori error estimates (3) as follows

$$||(I - P_h)F(U)||_{H_0^1} \leq C(h) \sup_{u \in U} ||f(u)||_{L^2}. \tag{6}$$

Thus, the former condition in (5) is validated as the inclusion relations of corresponding coefficient intervals, and the latter part can be confirmed by comparing two nonnegative real numbers which correspond to the radii of balls. In the actual computation, some iterative method is utilized for both part of $P_h F(U)$ and $(I - P_h)F(U)$.

Finite dimensional Newton's method Note that the verifiaction condition (5) is not applicable except that the concerned operator F is retractive around the fixed point. Therefore, in order to overcome this difficulty, we need some Newton-like method for (4). Thus, we define the nonlinear operator N_h with an approximate solution $\widehat{u}_h$ by

$$N_h(u) := P_h u - [P_h - P_h A'(\widehat{u}_h)]_h^{-1}(P_h u - P_h F(u)),$$

where $A'(\widehat{u}_h) \equiv (-\Delta)^{-1} f'(\widehat{u}_h)$ and $'$ means the Fréchet derivative of f at $\widehat{u}_h$. Here, $[P_h - P_h A'(\widehat{u}_h)]_h^{-1}$ denotes the inverse on S_h of the restriction operator $(P_h - P_h A'(\widehat{u}_h))|_{S_h}$. The existence of such a finite dimensional inverse operator can be validated by the usual invertibility of the corresponding matrix. And we set

$$T(u) \quad := \quad N_h(u) + (I - P_h)F(u).$$

Then T is considered as the Newton-like operator for the former part of (4) but the simple iterative operator for the latter part. It can be seen that $u = T(u)$ is equivalent to $u = F(u)$, and the verification condition is presented similar as before.

Infinite dimensional Newton's mtheod By applying the verification principle to the linearized equation for the original problem (1), we can also realize an infinite dimensional Newton-like mtheod.
We now assume that the linearlized equation at $\widehat{u}_h$ is written as

$$\begin{cases} \mathscr{L}u := -\Delta u + b \cdot \nabla u + cu = \psi, & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Here, assume $b \in W_1^\infty(\Omega)^n,\ c \in L^\infty(\Omega),\ \psi \in L^2(\Omega)$. Let ρ be an approximate operator norm for $\mathscr{L}^{-1}$, which can be computed as the corresponding matrix norm. Setting the constants as $C_{\operatorname{div} b} := \|\operatorname{div} b\|_{L^\infty(\Omega)}, \quad C_b := \left(\sum_{i=1}^n \|b_i\|_{L^\infty(\Omega)}^2\right)^{1/2}, \quad C_c := \|c\|_{L^\infty(\Omega)}$, and let $C_1 := C_p C_{\operatorname{div} b} + C_b, \quad C_2 := C_p C_c, \quad C_3 := C_b + C_p C_c, \quad C_4 := C_b + C(h) C_c$, where C_p is a Poincaré constant on Ω. Then, we have the following invertibility condition for $\mathscr{L}$ in (7).

Theorem 1.1. *If*

$$\kappa \equiv C(h)\big(\rho C_3 (C_1 + C_2) C(h) + C_4\big) < 1, \tag{8}$$

then the oprator $\mathscr{L}$ in (7) *is invertible. Here, $C(h)$ is the same constant in* (3).

By using this result we derive a verifiaction condition for the solution of the problem (1) to apply the infinite dimensional Newton-like method. On the other kind of verification methods for elliptic problems, refer [4, 5] and so on.

2 Evolutional problems

We can extend the arguments in the previous section to the following nonlinear intial boundary value problems of parabolic type.

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u &= f(x,t,u), \quad (x,t) \in \Omega \times J, \\ u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times J, \\ u(x,0) &= 0, \quad x \in \Omega, \end{cases} \tag{9}$$

where $x \in \Omega \subset R^d$: a bounded convex domain, $t \in J := (0,T) \subset R$: a bounded interval for a fixed T, and $\nu \in R$: a positive constant. We assume that f is a continuous map from $L^2(J;H_0^1(\Omega))$ into $L^2(J;L^2(\Omega))$, and, for each bounded subset U in $L^2(J;H_0^1(\Omega))$, the image of U by f is also bounded in $L^2(J;L^2(\Omega))$.

By using an appropriate approximate solution $u_h^k \in H^1(J;L^2(\Omega)) \cap L^2(J;H_0^1(\Omega) \cap H^2(\Omega))$ and setting $u \equiv w + u_h^k$, the original problem (9) can be rewritten in the following residual form

$$\begin{cases} \mathscr{L}_t w = g(w) & \text{in } \Omega \times J, \quad (10a) \\ w(x,t) = 0 & \text{on } \partial\Omega \times J, \quad (10b) \\ w(x,0) = 0 & \text{in } \Omega, \quad (10c) \end{cases}$$

where $\mathscr{L}_t := \frac{\partial}{\partial t} - \nu\Delta - f'(u_h^k)$. Here, $f'(u_h^k)$ stands for a Fréchet derivative of f at u_h^k. And $g(w) \equiv f(x,t,w+u_h^k,\nabla(w+u_h^k)) - \frac{\partial u_h^k}{\partial t} + \nu\Delta u_h^k - f'(u_h^k)w$.
In order to consider the existence of a solution w of (10), for any $\alpha > 0$, we define the candidate set by

$$W_\alpha := \{w \in L^2(J;H_0^1(\Omega));\ \|w\|_{L^2(J;H_0^1(\Omega))} \le \alpha\}. \tag{11}$$

If we find a constant $C_{\mathscr{L}_t^{-1}}$ satisfying

$$\|\mathscr{L}_t{}^{-1}\|_{\mathscr{L}\left(L^2(J;L^2(\Omega)),L^2(J;H_0^1(\Omega))\right)} \le C_{\mathscr{L}_t^{-1}}, \tag{12}$$

then, we have

$$\|\mathscr{L}_t{}^{-1} g(W_\alpha)\|_{L^2(J;H_0^1(\Omega))} \le C_{\mathscr{L}_t^{-1}} \sup_{w\in W_\alpha} \|g(w)\|_{L^2(J;L^2(\Omega))}.$$

Therefore, by the Schauder fixed point theorem we obtain the following existential condition of a solution $w \in W_\alpha$ to (10),

$$C_{\mathscr{L}_t^{-1}} \sup_{w\in W_\alpha} \|g(w)\|_{L^2(J;L^2(\Omega))} \le \alpha. \tag{13}$$

This clearly implies a Newton-type verification condition of solutions for the problem (9).

Note that usually $\mathscr{L}_t$ is written of the form

$$\mathscr{L}_t w \equiv \frac{\partial}{\partial t} w - \nu\Delta w + (b\cdot\nabla)w + cw, \tag{14}$$

where b and c are L^∞ functions on $\Omega\times J$. Hence we now consider the linear problems:

$$\begin{cases} \mathscr{L}_t w = q & \text{in } \Omega\times J, & (15a)\\ w(x,t) = 0 & \text{on } \partial\Omega\times J, & (15b)\\ w(x,0) = 0 & \text{in } \Omega, & (15c)\end{cases}$$

where the right-hand side q of (15a) means a given function in x and t. Thus, it is important and essential for our purpose to find a constant $C_{\mathscr{L}_t^{-1}}$ satisfying the following a priori estimates of solution w to (15)

$$\|w\|_{L^2(J;H_0^1(\Omega))} \quad \le \quad C_{\mathscr{L}_t^{-1}} \|q\|_{L^2(J;L^2(\Omega))},$$

which also implies that (12) holds for this constant $C_{\mathscr{L}_t^{-1}}$.

Thus, the method in the previous section can also, in principle, be applied to the verification of solutions of this problem. In such applications, the simple linear problem which corresponds to the Poisson equation in the elliptic case is as follows:

$$\begin{cases} \dfrac{\partial\phi}{\partial t} - \Delta\phi &=& g & (x,t)\in\Omega\times J,\\ \phi(x,t) &=& 0, & (x,t)\in\partial\Omega\times J,\\ \phi(x,0) &=& 0, & x\in\Omega,\end{cases} \tag{16}$$

where g is a known function. In [3], by using a full-discrete finite element approximation, we derived the constructive a priori error estimates of the form

$$\left\|\phi - \phi_h^k\right\|_{L^2\left(J;H_0^1(\Omega)\right)} \le C(h,k)\,\|g\|_{L^2\left(J;L^2(\Omega)\right)}, \tag{17}$$

where ϕ_h^k is a full-discerete approximation for the solution of (16) with mesh sizes h in space and k in time. The method uses a full-discrete numerical scheme which is based on an interpolation in time by using the fundamental solution for spatial discretization of (16). It is shown that, if we take $k = h^2$, the above constant $C(h,k)$ can be numerically estimated as $C(h,k) \approx O(h)$ as well as the corresponding L^2 estimates are order $O(h^2)$.
Thus, we obtain the constructive estimates of the constant $C_{\mathscr{L}_t^{-1}}$ by the arguments in [1]. We will show some nmerical examples for the computaion of $C_{\mathscr{L}_t^{-1}}$ and verification results for some prototype nonlinear problems in the talk.

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