Fast verified solutions of sparse linear systems

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To solve linear systems is ubiquitous since it is one of the basic and significant tasks in scientific computing. Floating-point arithmetic is widely used for this purpose. Since it uses finite precision arithmetic/numbers, rounding errors are included in computed results. To guarantee the accuracy of the results, there are methods so-called verified numerical computations based on interval arithmetic. Excellent overviews can be found in [6] and references cited therein.

Let A be a real $n \times n$ matrix, and b a real n-vector. Let $\kappa(A) = \|A\| \cdot \|A^{-1}\|$ be the condition number of A, where $\|\cdot\|$ stands for some matrix norm. Throughout the talk we assume for simplicity that IEEE standard 754 binary64 (formerly, double precision) floating-point arithmetic is used. Let $\mathbf{u}$ denote the rounding error unit of floating-point arithmetic, which is equal to 2^{-53}.

We are concerned with practically proving the nonsingularity of A (if A is nonsingular) and then obtaining a forward error bound of an approximate solution $\widetilde{x}$ of a linear system $Ax = b$ to the exact solution $x^* = A^{-1}b$ such that $|x_i^* - \widetilde{x}_i| \le \epsilon_i$ for $1 \le i \le n$ by the use of verified numerical computations. For this purpose estimating $\|A^{-1}\|$ is essential for some matrix norm.

For dense linear systems there are several efficient methods for this purpose (e.g. [1,4]). For sparse systems things are much different; Fast and efficient verification for large sparse linear systems is still difficult in terms of both computational complexity and memory requirements except a few cases where it is known in advance or to be proved that A belongs to a certain special matrix class, e.g. diagonally dominant and M-matrix (see, e.g. [3]). Moreover, a super-fast verification method proposed in [7] is applied to the case where A is sparse, symmetric and positive definite. However, to our knowledge, few methods are known in case of A being a general sparse matrix except methods by

Rump [5]. Thus the verification for sparse systems of linear (interval) equations is known as one of the important open problems posed by Neumaier in *Grand Challenges and Scientific Standards in Interval Analysis* [2]. Moreover, Rump [6] formulated the following challenge:

> Derive a verification algorithm which computes an inclusion of the solution of a linear system with a general symmetric sparse matrix of dimension 10000 with condition number 10^{10} in IEEE 754 double precision, and which is no more than 10 times slower than the best numerical algorithm for that problem.

In the present talk we try to partially solve the problem for symmetric but not necessarily positive definite input matrices, and also to a certain extent for nonsymmetric matrices. Namely, we assume that A is large, e.g. $n \geq 10000$, and sparse, possibly $\kappa(A) > 1/\sqrt{\mathbf{u}}$.

We survey some existing verification methods for sparse linear systems. After that, we propose new verification methods. Numerical results are also presented.

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