Computer assisted proof for existence of solutions to a system of elliptic partial differential equations

Kouta Sekine[1], Akitoshi Takayasu[2] and Shin'ichi Oishi[2,3]

[1] Graduate School of Fundamental Science and Engineering,
Waseda University
3–4–1 Okubo, Shinjuku-ku, Tokyo 169–8555, Japan

[2] Department of Applied Mathematics, Faculty of Science and Engineering,
Waseda University
3–4–1 Okubo, Shinjuku-ku, Tokyo 169–8555, Japan

[3] JST, CREST

s115100710@akane.waseda.jp

Keywords: Computer assisted proof, FitzHugh-Nagumo model

1 Introduction

We are concerned with a computer-assisted proof method for existence and local uniqueness of solutions to elliptic systems:

$$\begin{cases} -\varepsilon^2\Delta u = f(u) - \delta v & \text{in } \Omega, \\ -\Delta v = u - \gamma v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded polygonal domain with arbitrary shape in $\mathbb{R}^2$. $\varepsilon \neq 0$, γ and δ are real parameters. A mapping $f : H^1_0(\Omega) \to L^2(\Omega)$ is assumed to be Fréchet differentiable.

The system (1) is derived from the FitzHugh-Nagumo model. The original one-dimensional parabolic differential equation, which is called FitzHugh-Nagumo equation, was derived to serve as a prototype simplification of nerve conduction equations. The system (1) has been well studied from theoretical and numerical sides. A numerical verification theory for (1) on bounded convex domain has been proposed by Y. Watanabe [1].

The aim of this talk is to treat a numerical verification method of (1) on bounded nonconvex domains using Plum's Newton-Kantorovich like theorem [2]. On the nonconvex domain, calculating residual norm is one of the most important tasks because exact solutions u^* and v^* of (1) do not have H^2-regularity, respectively. In [3], A. Takayasu, X. Liu and S. Oishi have presented how to derive a residual norm using the Raviart-Thomas mixed finite element on a bounded polygonal domain. In this talk, we present the method of calculating for residual norm including a solution operator based on the Raviart-Thomas mixed finite element. Detailed proofs will be presented.

2 Numerical result

Let us consider the following Dirichlet boundary value problem of a system of nonlinear elliptic partial differential equations:

$$\begin{cases} -\Delta u = 100(u - u^3 - v) & \text{in } \Omega, \\ -\Delta v = u + 1.2v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is is a bounded nonconvex polygonal domain whose vertices are given by

$$\{(0,0), (0.2,0), (0.2,0.4), (0.8,0.4), (0.8,0), (1,0), (1,0.5), (0.5,1), (0.1,0)\}.$$

In Figure 1, we show approximate solutions. Verification result for (2) on Ω are given in Table 1. From Table 1, we succeed to prove the existence and local uniqueness of solutions which are located in neighborhood of these approximate solutions.

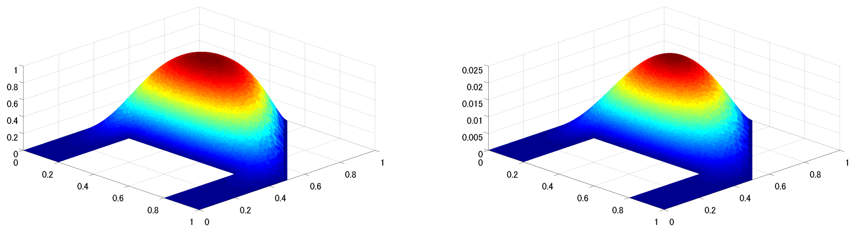

Figure 1: Approximate solution $\hat{u}$ (left) and $\hat{v}$ (right) of (2).

Table 1: Verification results.

Residual norm	$\|u^* - \hat{u}\|_{H_0^1}$	$\|v^* - \hat{v}\|_{H_0^1}$
1.244×10^{-2}	3.210×10^{-2}	9.060×10^{-4}

References:

[1] Y. Watamabe, A numerical verification method for two-coupled elliptic partial differential equations, *DMV Jahresbericht:*, JB.110, Heft 1, pp. 19-54, 2008.

[2] M. Plum, Existence and Multiplicity Proofs for Semilinear Elliptic Boundary Value Problems by Computer Assistance, *Japan J. Indust. Appl. Math.*, Vol.26, pp. 419-442, 2009.

[3] A. Takayasu, X. Liu, and S. Oishi, Verified computations to semilinear elliptic boundary value problems on arbitrary polygonal domains, *NOLTA*, Vol.4, pp. 24-61, 2013.