A method of verified computations for nonlinear homogeneous heat equations,

Part I: Enclosure of semidiscrete approximate solution for space variable

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Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. In this talk, we consider the following nonlinear heat equations of the form:

$$\begin{cases} \partial_t u = \Delta u + f(u) & \text{in } (0,\infty)\times\Omega, \\ u|_{\partial\Omega} = 0 & \text{in } (0,\infty), \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where $f : L^\infty((0,\infty); H_0^1(\Omega)) \to L^\infty((0,\infty); L^2(\Omega))$ be a nonlinear mapping. We assume the Fréchet differentiable of f with respect to u on the spatial direction. Furthermore, let $u_0 \in D(\Delta)$ be a given initial function. We will introduce a method of verified computations for the equation (1). Our method is based on an approximate solution that is numerically calculated by FEM and Backward Euler scheme [1,2,3]. Let $n \in \mathbb{N}$ be a fixed natural number. We divide the time: $0 = t_0 < t_1 < \cdots < t_n < \infty$. For $k = 1, 2, ..., n$, we define $T_k = (t_{k-1}, t_k]$ and $T = \bigcup T_k$. We can compute $\hat{u}_k \approx u(t_k)$ using the scheme established in [3]. Then we construct an approximate solution of (1), which is denoted by $\omega(t) \in L^\infty(T; H_0^1(\Omega))$:

$$\omega(t) := \sum_{k=1}^{n} \hat{u}_k \phi_k(t),\ t \in T,$$

where $\phi_k(t)$ is piecewise linear Lagrange basis on each T_k defined by

$$\phi_k(t) := \begin{cases} \dfrac{t - t_{k-1}}{t_k - t_{k-1}}, & t \in T_k, \\ \dfrac{t_{k+1} - t}{t_{k+1} - t_k}, & t \in T_{k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

In this part, we introduce how to verify a computable error bound ρ of

$$\|u - \omega\|_{L^\infty(T;H_0^1(\Omega))} \le \rho.$$

Namely, the existence and local uniqueness of $u(t)$ is shown in the ball:

$$B(\omega, \rho) := \{v \in L^\infty(T; H_0^1(\Omega)) : \|v - \omega\|_{L^\infty(T;H_0^1(\Omega))} \le \rho\}.$$

The feature of our method is to use ideal approximation: $\bar{u}(t) \in W^{1,1}(T; H_0^1(\Omega))$ defined by

$$\bar{u}(t) := \sum_{k=1}^{n} u_k \phi_k(t), \ t \in T,$$

where $u_k \in V$ satisfies

$$\tau_k^{-1}(u_k - u_{k-1}, v)_{L^2} + (\nabla u_k, \nabla v)_{L^2} = (f(u_k), v)_{L^2}, \ \forall v \in V.$$

Since $W^{1,1}(T; H_0^1(\Omega)) \hookrightarrow L^\infty(T; H_0^1(\Omega))$, we have

$$\begin{aligned} & \|u(t) - \omega(t)\|_{L^\infty(T;H_0^1(\Omega))} \\ = \ & \|u(t) - \bar{u}(t) + \bar{u}(t) - \omega(t)\|_{L^\infty(T;H_0^1(\Omega))} \\ \le \ & \|u(t) - \bar{u}(t)\|_{L^\infty(T;H_0^1(\Omega))} + \|\bar{u}(t) - \omega(t)\|_{L^\infty(T;H_0^1(\Omega))} \end{aligned}$$

Then our method is divided into two parts. First, we rigorously construct the ideal approximation $\bar{u}(t)$ on the basis of $\omega(t)$ using the framework of verified computations for elliptic equations. Next, the existence and local uniqueness of $u(t)$ is validated with computer-assistance depending on semigroup theory.

We first sketch our method briefly and explain how to construct the ideal approximation $\bar{u}(t)$ rigorously in the part I. In part II, on the basis of Banach's fixed point theorem, the existence and local uniqueness of $u(t)$ is proved via semigroup theory.

References:

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