Numerical Verification for Stationary Solutions to the Allen-Cahn Equation.

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To derive stationary solutions to the Allen-Cahn equation, we try to solve the following equation:

$$\begin{cases} -\varepsilon^2 \Delta u = u - u^3 & \text{in}\Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on}\partial\Omega, \end{cases} \tag{1}$$

where ε is a given positive number and Ω is a square domain $(0,1)^2$. The Allen-Cahn equation has various solutions, which form attractive patterns, especially when ε is small. For example, the following Fig. 1 shows some solutions in the case of $\varepsilon = 0.03$ (this is just a mere part).

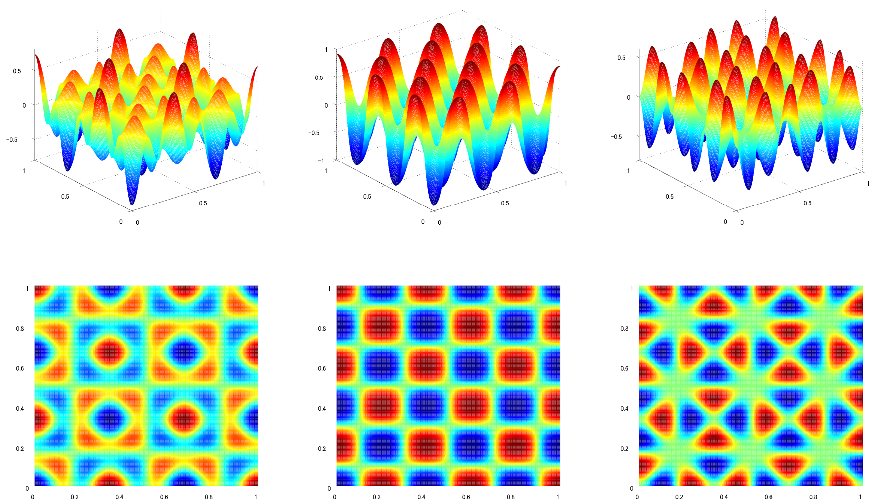

Fig. 1: Some solutions to (1) composed by sparse basis with $\epsilon = 0.03$.

Here, we set $V = H^1(\Omega)$ and denote the dual space of V by V^*. Defining operator $\mathcal{F} : V \to V^*$ by

$$\langle \mathcal{F}(u), v \rangle := (\nabla u, \nabla v)_{L^2} - \varepsilon^{-2} \left(u - u^3, v\right)_{L^2}, \forall v \in V,$$

the equation (1) can be transformed into the equation

$$\mathcal{F}(u) = 0 \text{ in } V^*.$$

We derived the approximate solutions to this equation with spectral method and verified these solutions using Newton-Kantorovich's theorem (the verification method with this theorem summarized in [1]). One of the most important thing for verification is how to estimate the norm of inverse of linearized operator $\|\mathcal{F}'[\hat{u}]^{-1}\|$, where $\hat{u} \in V$ is an approximate solution and $\mathcal{F}'[\hat{u}]$ is the Fréchet derivative of $\mathcal{F}$ at $\hat{u}$. We estimated the operator norm using the theorem in [2] based on Liu-Oishi's theorem [3] which is an effective theorem to evaluate eigenvalues of the Laplace operator on arbitrary polygonal domains.

Since a small ε makes solutions to (1) singular, a more accurate basis becomes necessary to obtain an appropriate approximate solution for small ε. Of course, numerical verification also becomes difficult when ε is small at least using the usual basis.

We observed that there are many approximate solutions which may have not specific frequency components periodically (we call this type of solution "sparse solution"). Therefore, we can fast derive and verify an approximation of sparse solutions by removing the basis functions corresponding to the frequency components expected that the solutions do not have (we call this type of basis "sparse basis"). Unfortunately, there is no evidence that an appropriate approximate solution is obtained using sparse basis. Indeed, approximate solutions which are not obtained with the usual basis are often obtained a with sparse basis in our experience. For this reason, verification for solution's existence is indispensable when we use sparse basis. In Fig. 1, some verified solutions composed by sparse basis are displayed.

In this talk, a consideration about behavior of solutions to (1) with numerical verification also will be performed.

References:

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