# A comparison of computer-assisted proofs for the Kolmogorov problem

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Consider the Navier-Stokes equations:

$$u_t + uu_x + vu_y = \nu\Delta u - \frac{1}{\rho} p_x + \gamma \sin\left(\frac{\pi y}{b}\right), \tag{1}$$

$$v_t + uv_x + vv_y = \nu\Delta v - \frac{1}{\rho} p_y, \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

where $(u, v)$, $\rho$, $p$ and $\nu$ are velocity vector, mass density, pressure and kinematic viscosity, respectively and $\gamma$ is a constant representing the strength of the sinusoidal outer force. Also $*_\xi := \partial/\partial\xi(\xi = t, x, y)$ and $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$. The flow region is a rectangle $[-a, a] \times [-b, b]$ and the periodic boundary conditions are imposed in both directions. The aspect ratio is denoted by $\alpha := b/a$.

The equations (1–3) describe the Navier-Stokes flows in a two-dimensional flat torus under a special driving force proposed by Kolmogorov [1], [2] and have a basic solution which is written as $(u, v, p) = (k \sin(\pi y/b), 0, d)$, where $k := b^2\gamma/(\pi^2\nu)$ and $d$ is any constant. It is known that non-trivial solutions bifurcate from the basic solution at a certain Reynolds number, which is defined below, if and only if $0 < \alpha < 1$ [1]. Okamoto-Shoji [2] computed numerically bifurcation diagrams with the Reynolds number as a bifurcation parameter varying the aspect ratio as a splitting parameter. They also strongly suggested stability of the first bifurcating solutions for all $0 < \alpha < 1$. Nagatou [3] took an another approach to this stability problem by employing the theory of verified computation and showed that the stability of the first bifurcating solutions is mathematical rigorously assured for the cases of $\alpha = 0.4$, 0.7 and 0.8.

In the previous paper [4], we proposed a method to prove the existence and the local uniqueness of the steady-state solutions of the Navier-Stokes equations (1–3) for a given Reynolds number and aspect ratio by a computer-assisted proof with some verified results. It was also the first theoretical results to the non-trivial solutions of the equations (1–3).

The aim of this lecture is to apply our another verification method: FN-Int [5] to prove the existence of the steady-state solutions of problem (1–3).

Introducing the stream function $\phi$ satisfying $u = \phi_y$ and $v = -\phi_x$ so that $u_x + v_y = 0$, the equations (1–3) can be rewritten as

$$(\Delta\phi)_t - \nu\Delta^2\phi - J(\phi, \Delta\phi) = \frac{\gamma\pi}{b} \cos\left(\frac{\pi y}{b}\right) \tag{4}$$

by cross-differentiating equations (1) and (2) and eliminating the pressure $p$. Here $J$ is a bilinear form defined by

$$J(u,v) := u_x v_y - u_y v_x. \tag{5}$$

The equation (4) is nondimensionalized by using change of variables

$$(x',y') = \left(\frac{\pi x}{b}, \frac{\pi y}{b}\right), \quad t' = \frac{\gamma b}{\nu\pi}t, \quad \phi'(t',x',y') = \frac{\nu\pi^3}{\gamma b^3}\phi(t,x,y)$$

and the Reynolds number $R := \dfrac{\gamma b^3}{\nu^2\pi^3}$. After dropping the primes, an equation

$$(\Delta\phi)_t - \frac{1}{R}\Delta^2\phi - J(\phi,\Delta\phi) = \frac{1}{R}\cos(y) \tag{6}$$

is obtained.

Now let a rectangle region $\mathbf{T}_\alpha$ be

$$\mathbf{T}_\alpha := \left(-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right) \times (-\pi,\pi)$$

with aspect ratio $0 < \alpha < 1$. We shall find *steady-state solutions* on $\mathbf{T}_\alpha$, where $(\Delta\phi)_t$ is equated to 0 in equation (6) in the region $\mathbf{T}_\alpha$, namely consider the following nonlinear problem:

$$\Delta^2\phi = -R\,J(\phi,\Delta\phi) - \cos(y) \ \text{ in } \ \mathbf{T}_\alpha. \tag{7}$$

Assume that the stream function $\phi$ is periodic in $x$ and $y$, and the symmetric condition $\phi(x,y) = \phi(-x,-y)$ [3] as well as the normalization $\int_\Omega \psi\, dxdy = 0$. Then the equation (7) has a trivial solution $\phi = -\cos(y)$ for any $R > 0$. We will verify the existence of non-trivial solutions by a computer.

¿From the assumptions of $\psi$ imposed above, we define function space $X^k$ $(k \geq 0)$ by the closure in $H^k(\mathbf{T}_\alpha)$ of the linear hull of all functions $\cos(m\alpha x + ny)$ $(m \in \mathbb{N}_0,\ n \in \mathbb{Z},\ (m,n) \neq (0,0))$. Especially we define

$$X := X^3.$$

For each $\psi \in X^k$ can be represented by

$$\psi = \sum_{(m,n)\in Q} A_{mn}\cos(m\alpha x + ny), \qquad A_{mn} \in \mathbb{R},$$

where

$$Q := \left\{ (m,n) \in \mathbb{Z}\times\mathbb{Z} \;\middle|\; \begin{array}{l} \text{“}m = 0 \ \text{ and } \ 1 \leq n \leq \infty\text{''} \ \text{ or} \\ \text{“}1 \leq m \leq \infty \ \text{ and } \ -\infty \leq n \leq \infty\text{''} \end{array} \right\}. \tag{8}$$

Let $X_N$ be the finite-dimensional subspace of $X$, which depends on a non-negative integer parameter $N$, defined by

$$X_N := \left\{ \sum_{(m,n)\in Q_N} A_{mn}\cos(m\alpha x + ny) \ \middle|\ A_{mn} \in \mathbb{R} \right\}, \tag{9}$$

where

$$Q_N := \left\{ (m,n) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; \begin{array}{l} \text{“}m = 0 \;\text{ and }\; 1 \le n \le N'' \;\text{ or} \\ \text{“}1 \le m \le N \;\text{ and }\; -N \le n \le N'' \end{array} \right\}. \tag{10}$$

Define the projection $X \to X_N$ by the $N$-th truncation of Fourier expansion. Note that by the orthogonality of the basis $P_N$ coinsides with usual $H_0^2$-projection:

$$(\, \Delta(\psi - P_N\psi), \Delta\psi_N \,)_{L^2} = 0, \qquad \forall \psi_N \in X_N. \tag{11}$$

For each $g \in X^0$ let $\xi \in X^4$ the solution of $\Delta^2\xi = g$, then an a priori error estimate:

$$\|\xi - P_N\xi\|_X \le C_5 \|g\|_{L^2}$$

holds, where

$$C_5 = \frac{1}{\alpha(N+1)}. \tag{12}$$

Now for fixed approximate solution $\phi_N \in X_N$ of (7), setting

$$\phi = \phi_N + \psi, \tag{13}$$

and substituting (13) to (7), we obtain a residual equation

$$\Delta^2\psi = -R\; J(\phi_N + \psi, \Delta\phi_N + \Delta\psi) - \cos(y) - \Delta^2\phi_N \;\text{ in }\; \Omega. \tag{14}$$

Denote the right hand side of (14) by

$$f(\psi) := -R\; J(\phi_N + \psi, \Delta\phi_N + \Delta\psi) - \cos(y) - \Delta^2\phi_N, \tag{15}$$

$f : X \to X^0$ is continuous and maps any bounded set of $X$ to a bounded set of $X^0$.

Moreover, for each $g \in X^0$, $\Delta^2\xi = g$ has a unique solution $\xi \in X^4$. By denoting this mapping with embedding $X^4 \hookrightarrow X$ by

$$\Delta^{-2} : X^0 \longrightarrow X,$$

and

$$F := \Delta^{-2} f : X \longrightarrow X,$$

$F$ becames compact operator and problem (14) is equivalent to a fixed-point equation

$$\psi = F(\psi) \tag{16}$$

in $X$. Therefore we can apply our verification algorithm FN-Int [5]. We will report on some comparisons for computer-assisted proofs and show the effectiveness of FN-Int in the Kolmogorov problem (7).

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