Numerical verification methods for limit cycles in dynamical systems

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1 Previous works

Poincaré map is a general tool to treat limit cycles in dynamical systems. In order to prove existence of a limit cycle by validated computation, Zgliczyński verified existence of a fixed point of a Poincaré map using a fixed point theorem[5]. However it was not an easy work to specify 'first return time' T_s, a time period between an initial point $\mathbf{x}_0$ on the Poincaré section Γ and $\mathbf{x}_1 := \varphi(T_s, \mathbf{x}_0) \in \Gamma$, where $\varphi(t, \mathbf{x}_0)$ denotes a point on the trajectory from $\mathbf{x}_0$ at time t. Of course one have to verify that there is no point $\varphi(t, \mathbf{x}_0) \in \Gamma$ for any $t \in (0, T_s)$. Zgliczyński proposed a way to handle the situation and showed numerical examples to appeal effectiveness of his method.

Hereafter we propose another way in which one has not to construct a Poincaré map any longer.

2 Problem

We treat an autonomous dynamical system described by an ordinary differential equation as follows.

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}), \tag{1}$$

where $\mathbf{u}$ is a vector function in $D \subset \mathbf{R}^n$ and $\mathbf{f}(\mathbf{u})$ is a mapping from D to $\mathbf{R}^n$, which is differentiable with respect to $\mathbf{u}$.

Suppose that some approximate computation suggests that the above system may have a periodic orbit with an initial value $\tilde{\mathbf{u}}_0$ and a time period $\tilde{T}$, and such initial values are not distributed continuously. Our aim is to verify existence of a limit cycle including a point within a neighborhood of $\tilde{\mathbf{u}}_0$.

We translate the axes such that $\tilde{\mathbf{u}}_0$ should be an origin of new axes, and take a plane Γ with its unit normal vector $\mathbf{n}_\Gamma$. The plane Γ should include the new origin and may play a roll of Poincaré section.

3 Our first idea

Our first idea is using a projection P_w which maps a point $\mathbf{u}$ to a point $\mathbf{w}$ on Γ such that the vector $\mathbf{u} - \mathbf{w}$ should be parallel to the normal $\mathbf{n}_\Gamma$. We extend the phase space R^n to R^{n+1} with $\mathbf{z} = (t, \mathbf{u})^T$.

Let $\mathbf{z}_0 = (T_0, \mathbf{w}_0)^T$ be a pair of an approximate time period T_0 of a limit cycle and an approximate initial point $\mathbf{w}_0 \in \Gamma$. Calculate a new approximate period T_1 by some Newton-like method using T_0 and $\mathbf{w}_0$, and put $\mathbf{w}_1 := \varphi(T_1, \mathbf{w}_0)$. Define an operator Q as an operator on R^{n+1} which maps $\mathbf{z}_0 = (T_0, \mathbf{w}_0)^T$ to $\mathbf{z}_1 = (T_1, \mathbf{w}_1)^T$. One may expect that an iterative method might reach a fixed point of a Poincaré map applying Newton type iteration to an operator $I - Q$, where I is the identity operator. But this is not true since the Jacobi matrix of $I - Q$ has an eigenvalue almost equal to 0. In order to handle this situation, we introduce an operator P_Γ which maps $\mathbf{z} = (t, \mathbf{u})^T$ to $P_\Gamma \mathbf{z} = (t, P_w \mathbf{u})^T$ and apply a Newton type iteration to $I - P_\Gamma Q$.

This device works well since the Jacobi matrix of $I - P_\Gamma Q$ has no zero-eigenvalue in many cases. We published a paper on our method in Japanese[1] together with numerical verification process and some numerical examples.

Note that we do not construct any Poincaré map and do not suffer from computation of the first return times.

4 Revision

Professor Matsuo in Tokyo University and his student Kaigaishi pointed out that our method can be simplified and the simplified version concerns Newton-Raphson-Mees method[3] which is known as a useful tool to detect periodic orbits [2]. They says Newton-Raphson-Mees method can be considered as a variation of our method.

We omit the details of how to simplify our method, which will be explained in our talk. Hereafter we just describe the simplified version.

Let H be an operator from R^{n+1} to R^n for $\mathbf{z} = (t, \mathbf{w})^T$ as follows.

$$H(\mathbf{z}) = \varphi(t, \mathbf{w}) - \mathbf{w}.$$

Note that a zero point $\mathbf{z}^* = (T^*, \mathbf{w}^*)^T$ of H, namely $H(\mathbf{z}^*) = \mathbf{0}$, should be a fixed point of a Poincaré map. Therefore if we verify that there is a zero point of the operator H within some small area $([T], [\mathbf{w}])^T$, then it is proven that there is a closed orbit including a point in $[\mathbf{w}]$ with a time period $T \in [T]$. Here the square brackets $[\cdot]$ denote interval values.

To find a zero point of H we need an additional condition, and adopt

$$\mathbf{n}_\Gamma^T \varphi(T^*, \mathbf{w}^*) \quad = \quad 0, \tag{2}$$

which means that we will seek a zero point on the plane Γ.

Summing up the above, we define the operator K as follows.

$$K(\mathbf{z}) \quad = \quad \begin{pmatrix} \mathbf{n}_\Gamma^T \varphi(t, \mathbf{w}) \\ H(\mathbf{z}) \end{pmatrix},$$

and we will solve the equation

$$K(\mathbf{z}) = \mathbf{0}$$

with validated computation. In order to apply a Newton type iteration, Jacobi matrix DK of the operator K is necessary. Note that DK is simply represented as

$$DK(\mathbf{z}) = \begin{pmatrix} 0 & \mathbf{n}_\Gamma^T \\ \mathbf{f}(\mathbf{z}) & D_\mathbf{w}\varphi(t,\mathbf{w}) \end{pmatrix},$$

where $D_\mathbf{w}\varphi(t,\mathbf{w})$ denotes Jacobi matrix of $\varphi(t,\mathbf{w})$ with respect to $\mathbf{w}$.

5 Newton Operators

We use two kinds of Newton type operators N_1 and N_2 to find zeros of the operator K. Let $[T]$ be an interval value of the time period T, $[\mathbf{w}]$ be an interval vector which contains points on the plane Γ, and define $[\mathbf{z}] = ([T],[\mathbf{w}])^T$ with its center vector $\hat{\mathbf{z}}$.

The operator N_1 is defined for the interval Newton method as follows.

$$N_1([\mathbf{z}]) = \hat{\mathbf{z}} - DK([\mathbf{z}])^{-1}K(\hat{\mathbf{z}}).$$

The operator N_2 is defined for the quasi Newton method as follows.

$$N_2([\mathbf{z}]) = [\mathbf{z}] - DK_a^{-1}K([\mathbf{z}]),$$

where DK_a denotes an approximation to $DK(\tilde{\mathbf{z}})$ for $\tilde{\mathbf{z}} = (\tilde{T},\tilde{\mathbf{u}})^T$ (Note that we put $\tilde{\mathbf{u}} = \mathbf{0}$ by translation of axes).

One may consider that the operator N_2 has an advantage over N_1 since N_2 has no interval matrix. However, we have to point out that subdistributive law for interval arithmetic may cause too much expansion on the radius of the interval value $[\mathbf{z}] - DK_a^{-1}K([\mathbf{z}])$. To avoid such expansion, the mean value form should be applied to the interval, and we have

$$N_2(\mathbf{z}) = DK_a^{-1}\{DK_a\hat{\mathbf{z}} - K(\hat{\mathbf{z}}) + (DK_a - DK([\mathbf{z}]))([\mathbf{z}] - \hat{\mathbf{z}})\}.$$

Then N_2 also has an interval matrix. There is a possibility that N_2 still has some advantage since it does not need the inverse of the interval matrix.

6 Verification of zero points

Take an interval vector $[\mathbf{Z}] = ([T],[\mathbf{W}])^T$ with a small radius, where $[\mathbf{W}] \subset \Gamma$. In many cases $[\mathbf{Z}]$ includes an approximate zero point $(\tilde{T},\mathbf{0})^T$. We iterate the following steps for N_i, $i = 1$ or $i = 2$.

(1) Check whether $N_i([\mathbf{Z}]) \subset [\mathbf{Z}]$ or not. If it holds then Brouwer's fixed point theorem guarantees the existence of a fixed point of N_i within $[\mathbf{Z}]$, which is a zero of the operator K.

(2) If it does not hold, we calculate a new candidate as

$$[\mathbf{Z}] \quad := \quad (1+\varepsilon)N_i([\mathbf{Z}]) - \varepsilon N_i([\mathbf{Z}]),$$

with a given small positive ε, and iterate furthermore.

Numerical experiments will be shown in our talk.

7 Remarks

- We have to compute $\varphi([T],[\mathbf{W}])$ by validated computation with interval arithmetic. Integration is carried out using Lohner method[4].
- The Jacobi matrix $DK([\mathbf{Z}])$ includes the Jacobi matrix $D_{\mathbf{w}}\varphi([T],[\mathbf{W}])$, which should be computed by validated computation. Integration is also carried out using Lohner method, and there is an efficient way to compute it together with $\varphi([T],[\mathbf{W}])$ simultaneously. This is described by Zgliczyński in [5].
- The advantage of our method is not to construct any Poincaré map, and not to confirm the first return times.
- Thanks to the advice from Professor Matsuo and his student Kaigaishi, our method turned to be given a very simple description and to be easy to understand. Use it!

References:

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